

BA'ZI KLASSIK MUAMMOLAR VA TAQSIMOTLAR

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Annotatsiya:

Maqolada Ba'zi muhim taqsimotlar (Bernulli, Binomial, Puasson Tekis, Normal, ko'rsatkichli) asosan Binomial taqsimot, Vandermond taqsimot va Nyorlund taqsimotlari keltirilgan yuzasida masalalar keltirilgan.

Kalit so'zlar: Ba'zi muhim taqsimotlar (Bernulli, Binomial taqsimot, Vandermond taqsimot va Nyorlund taqsimotlari)

Kirish

¹Bu maqolada biz Ba'zi muhim taqsimotlar (Bernulli, Binomial, Puasson Tekis, Normal, ko'rsatkichli) asosan Binomial taqsimot, Vandermond taqsimot va Nyorlund taqsimotlari haqida ma'lumotlarga ega bo'lishimiz kerak.

№ 1. Quyidagilarni ko'rsating

$$(a) \quad (x + y)^n = \sum_{k=0}^n C_n^k x^k y^{n-k} \quad (\text{Binomial taqsimot}),$$

$$(b) \quad (x + y)_n = \sum_{k=0}^n C_n^k (x)_k (y)_{n-k} \quad (\text{Vandermond taqsimot}),$$

$$(c) \quad [x + y]_n = \sum_{k=0}^n C_n^k [x]_k [y]_{n-k} \quad (\text{Nyorlund taqsimot}),$$

bu yerda

$$(x)_n = x(x-1)(x-2)\dots(x-n+1),$$

$$[x]_n = (x+1)(x+2)\dots(x+n-1).$$

¹ [[1]-73-93-betlarni qarang]

Yechim. (a) Agar $(x + y)^n$ ichidagi barcha qavslarni o'xshash hadlarni qatnashmasdan ochsak, $x^k y^{n-k}, k = 0, \dots, n$. ko'rinishdagi 2^n ta hadga ega bo'lamiz. Agar $x^k y^{n-k}$ ichidagi k ta qavsdan x hadni, qolganlaridan y ni olsak, $(x + y)^n$ hadi olinadi. $(x + y)^n$ ichida aynan k ta qavs C_n^k usullarida tanlanishi mumkin, ya'ni $(x + y)^n$ da qavslarni ochishda $x^k y^{n-k}$ lar soni bir xil bo'ladi.

(b) X to'plamning elementlaridan farqli $x, y \in N, x \geq n, y \geq n$. bo'lsin, x farqli elementlardan iborat X to'plam va y elementlardan iborat Y to'plamlar ko'rib chiqilsin. $X \cup Y$ uzunligi n dan tartiblangan to'plam elementlari soni. Bir tomondan, bu n bo'ylab $x + y$ ning tartibga solish soni. Boshqa tomondan, barcha to'plamlarni ulardagi X elementlar soniga ko'ra sinflarga bo'lish mumkin, bu elementlarning sonini k deb belgilaydi. Uzunligi k bo'lgan X tartiblangan elementlar to'plami soni $\binom{x}{k}$, Y dan $n - k$ uzunlik $\binom{y}{n-k}$ ga teng. X elementlar va Y elementlar o'rtasida tartibni saqlagan holda bu to'plamlarni aralashtirish usullari soni C_n^k .

Shunday qilib, har bir sinfdagi to'plamlar soni $C_n^k x_k y_{n-k}$ bo'lib, bu $x, y \in N, x \geq n, y \geq n$ uchun kerakli formulani isbotlaydi.

Agar $x, y \in R$ bo'lsa, biz Vandermonde taqsimotining chap va o'ng tomonlari x va y ko'phadlari ekanligini topamiz, ularning qiymatlari sanab o'tiladigan nuqtalar soniga to'g'ri keladi. Bu shuni anglatadiki, ko'phadlar bir xil tengdir.

(c) Agar qarama-qarshi belgi bilan x va y ni olsak va quyidagi

$$(-x)_n = \binom{x}{n} (-1)^n$$

dan, Norlundning o'ziga xosligi Vandermonde taqsimotidan kelib chiqadi.

№ 2. Ehtimoliy, kombinatoriyal, geometrik mulohazalardan foydalanish (masalan, qulay imkoniyatlar sonini yoki bir nuqtadan ikkinchisiga olib boradigan yo'llar sonini turli yo'llar bilan hisoblash) va boshqa mulohazalar (aytaylik, algebraik, masalan, x^n uchun koeffitsientlarni hisoblash va taqqoslashda va quyidagi tenglikda

$$(1 + x)^a (1 + x)^b = (1 + x)^{a+b},$$

Binomial koeffitsientlarning quyidagi xossalarini qaraymiz:²

$$(a) \quad 1 = C_n^0 < C_n^1 < \dots < C_n^{\lfloor n/2 \rfloor} = C_n^{\lceil n/2 \rceil} > \dots > C_n^{n-1} > C_n^n = 1$$

(bu yerda simmetriya va unimodallik xosasi

$$\lfloor x \rfloor = \max n \in Z : n \leq x, \quad \lceil x \rceil = \min n \in Z : n \geq x),$$

$$(b) \quad C_n^{k-1} + C_n^k = C_{n+1}^k \text{ -“Paskal uchburchagi”, “Paskal taqsimoti”},$$

$$(c) \quad C_n^k = C_{n-2}^k + 2C_{n-2}^{k-1} + C_{n-2}^{k-2},$$

$$(d) \quad \sum_{k=0}^n C_n^k = 2^n,$$

² [[3]-123-144-betlarni qarang] binomial taqsimot xossalari keltirilgan.

- (e)
$$\sum_{k=0}^n (C_n^k)^2 = C_{2n}^n,$$
- (f)
$$\sum_{k=0}^n 2^k C_n^k = 3^n,$$
- (g)
$$\sum_{k=1}^n k C_n^k = n 2^{n-1},$$
- (h)
$$\sum_{m=0}^k (-1)^{k-m} C_n^m = C_{n-1}^k, \quad n \geq k + 1,$$
- (i)
$$\sum_{k=0}^n k(k-1) C_n^k = n(n-1) 2^{n-2}, \quad n \geq 2,$$
- (j)
$$k C_n^k = n C_{n-1}^{k-1},$$
- (k)
$$\sum_{m=k}^n C_m^k = C_{n+1}^{k+1},$$
- (l)
$$C_n^m C_m^k = C_n^k C_{n-k}^{m-k}, \quad k \leq m \leq n,$$
- (m)
$$\sum_{m=0}^k C_{n+m}^m = C_{n+k+1}^k,$$
- (n)
$$\sum_{m=0}^k C_{n+1}^m = \sum_{m=0}^k 2^m C_{n-m}^{k-m}, \quad k \leq n,$$
- (a*)
$$C_{n-k}^k + C_{n-k-1}^{k-1} = \frac{n}{n-k} C_{n-k}^k, \quad k \leq n,$$
- (b*)
$$C_{n_1-n_2}^n = \sum_{k=0}^{n_1} C_{n_1}^k C_{n_2}^{n-k} \quad \text{-Vandermonde binomial taqsimoti,}$$
- (c*)
$$C_n^{k-1} C_n^{k+1} \leq (C_n^k)^2, \quad k \leq n-1,$$
- (d*)
$$C_{m+n}^n \leq \left(1 + \frac{m}{n}\right)^n \left(1 + \frac{n}{m}\right)^m, \quad \text{yoki ekvivalent} \quad \frac{(m+n)!}{(m+n)^{m+n}} \leq \frac{m!n!}{m^m n^n},$$
- (e*)
$$\frac{n^k}{k^k} \leq C_n^k \leq \frac{n^k}{k!},$$
- (f*)
$$\sum_{k=0}^n C_n^k (-1)^{n-k} 2^k = 1,$$
- (g*)
$$C_n^k = \sum_{m=k}^n (-1)^{n-m} C_n^m C_m^k 2^{m-k}, \quad 1 \leq k \leq n,$$

$$(h^*) \quad \sum_{k=0}^n (-1)^k (C_n^k)^2 = \begin{cases} (-1)^m \frac{(2m)!}{(m!)^2}, & \text{agar } n = 2m \\ 0, & \text{aks holda} \end{cases}$$

$$(i^*) \quad \sum_{k=0}^n (-1)^k (C_n^k)^3 = \begin{cases} (-1)^m \frac{(3m)!}{(m!)^3}, & \text{agar } n = 2m \\ 0, & \text{aks holda} \end{cases} \quad \text{-Dikson ayniyat}$$

$$(j^*) \quad \sum_{k=1}^n C_n^k \frac{(-1)^{k-1}}{k} = \sum_{k=1}^n \frac{1}{k},$$

$$(k^*) \quad \sum_{k=0}^n (-1)^{n-k} k^m C_n^k = \begin{cases} 0, & \text{agar } m < n, \\ n!, & \text{agar } m = n, \end{cases}$$

$$(l^*) \quad \sum_{k=0}^n C_{2(n-k)}^{m-k} C_{2k}^k = 2^{2n},$$

$$(m^*) \quad \sum_{k=1}^n (-1)^{k-1} C_n^k \sum_{m=1}^k \frac{1}{m} = \frac{1}{n},$$

$$(n^*) \quad \sum_{m=0}^k C_n^m C_m^k = 2^{n-k} C_n^k.$$

Yechim. Taqdimotning qisqaligi uchun biz dalillarda faqat algebraik mulohazalardan foydalanamiz.

(a), (b), (c), (j), (l), (a*), (c*), (e*) munosabatlari bevosita tekshirish orqali o'rnatiladi. Tenglik (d),

(f), (f*) kengayish $(x + y)^n$ dan turli xil tanlash x va y usullari ostida kelib chiqadi. Tenglikning

(g), (i) ikkala qismi mos ravishda teng $\left(1 + x^n\right)' \Big|_{x=1}$ va $\left(1 + x^n\right)'' \Big|_{x=1}$. Bundan tashqari,

munosabatlar (b*), (e), (h*) kengayishlarda mos ravishda x^n uchun k koeffitsientlarining tengligi sifatida $C_n^k = C_n^{n-k}$, xossasini hisobga olgan holda olinadi.

$$(1 + x)^{n_1} (1 + x)^{n_2} = (1 + x)^{n_1 + n_2},$$

$$(1 + x)^n (1 + x)^n = (1 + x)^{2n},$$

$$(x - 1)^n (x + 1)^n = (x^2 - 1)^n.$$

(h) Paskalning taqsimotidan $C_n^k = C_{n-1}^k + C_{n-1}^{k-1}$, ni olamiz, bu

$$C_n^k - C_n^{k-1} + C_n^{k-2} - \dots + (-1)^k = C_{n-1}^k + C_{n-1}^{k-1} - C_{n-1}^{k-1} - C_{n-1}^{k-2} + C_{n-1}^{k-3} - \dots = C_{n-1}^k.$$

ekanligini bildiradi.

(k) Paskal taqsimotidan va xossasidan foydalanib $C_k^k = C_{k+1}^{k+1}$, ni topamiz

$$\begin{aligned} C_{k+1}^{k+1} + C_{k+1}^k + C_{k+2}^k + \dots + C_n^k &= C_{k+2}^{k+1} + C_{k+2}^k + \dots + C_n^k = \\ &= C_{k+3}^{k+1} + C_{k+2}^k + \dots + C_n^k = C_{n+1}^{k+1}. \end{aligned}$$

(m) tenglik koeffitsientlarning tengligidan kelib chiqadi. x^n ning kengaytmalarida

$$\sum_{m=0}^k (1+x)^{m+n} = \frac{1}{x} \left(1+x^{n+k+1} - (1+x)^n \right).$$

(n) $\sum_{m=0}^k C_{n+1}^m = 2^{n+1} - \sum_{m=0}^{n-k} C_{n+1}^m$ ekanligini unutmang. Istalgan xususiyat kengayishida x^{n-k} uchun koeffitsientlarning tengligi quyidagicha

$$\begin{aligned} (1+x+x^2+\dots)(2^{n+1} - (1+x)^{n+1}) &= \frac{2^{n+1} - (1+x)^{n+1}}{1-x} = \\ &= \sum_{m=0}^n (1+x)^{n-m} 2^m. \end{aligned}$$

(d*) Bizda

$$\begin{aligned} C_{m+n}^m &= \frac{(m+n)!}{m!n!} = \left(\frac{1}{n} + \frac{m}{n} \right) \dots \left(1 + \frac{m}{n} \right) \leq \left(1 + \frac{m}{n} \right)^n \leq \\ &\leq \left(1 + \frac{m}{n} \right)^n \left(1 + \frac{n}{m} \right)^m \end{aligned}$$

bor.

(i*) $\sum_{k=0}^n (-1)^k (C_n^k)^3$ qiymat kengayishidagi $x^n y^n z^n$ koeffitsientiga ajraladi

$$(x-y)^n (y-z)^n (z-x)^n$$

$(x-y)^{n-1} (y-z)^{n-1} (z-x)^{n-1}$ ko'phad toq n uchun simmetrikdir, demak

$$(x-y)(y-z)(z-x) = x^2z - x^2y + y^2x - y^2z + z^2y - z^2x, \text{ dagi koeffitsient } x^n y^n z^n$$

nolga teng degan xulosaga kelish oson.

Endi n teng bo'lsin. Keling uni qo'yamiz

$$C_x^k = \frac{x(x-1)\dots(x-k+1)}{k!}.$$

$2n$ darajali ko'phadlarning quyidagi tengligini isbotlaymiz:

$$\sum_{k=0}^{2n} (-1)^k C_{2n}^k C_x^k C_x^{2n-k} = (-1)^n C_x^n C_{x+n}^n.$$

$x = 0, 1, \dots, n-1$ da ikkala tomon ham nolga teng, chunki $C_x^m = 0$ va $C_x^k = 0$ yoki

$C_x^{2n-k} = 0$. $x = n$ da birinchi ko'phad ikkinchi ko'phadning $x = n$ nuqtasidagi qiymatiga teng

nolga teng bo'lmagan bitta hadga ega bo'ladi. $x = -n, -n+1, \dots$ bizda $C_{x+n}^m = 0$ bor

$$f(k) = C_x^k C_x^{2n-k} = C_{|x|+k-1}^k C_{|x|+2n-k-1}^{n-k} = C_{|x|+k-1}^{|x|-1} C_{|x|+2n-k-1}^{|x|-1}$$

funksiya esa k (darajalari $2|x| - 2 < 2n$), da ko'phad, shuning uchun (k^*) xossasidan kelib chiqib, quyidagiga ega bo'lamiz.

$$\sum_{k=0}^{2n} (-1)^k C_{2n}^k f(k) = 0.$$

Shunday qilib, $2n$ darajali ko'phadlar $2n + 1$ nuqtaga to'g'ri keldi, shuning uchun ular bir-biriga teng. $x = 2n$ o'rnatib, biz Diksonning taqsimotini olamiz hatto n uchun ham.

(g^*) va (n^*) munosabatlar x uchun koeffitsientlarning tengligidir x^{n-k} mos ravishda kengayishlarda

$$(1+x)^n = \sum_{m=0}^n C_n^m (1+2x)^m (-x)^{n-m}$$

va

$$(1+2x)^n = \sum_{m=0}^n C_n^m (1+x)^m x^{n-m}.$$

(j^*) Bizda

$$\sum_{k=1}^n C_n^k \frac{(-1)^{k-1}}{k} = \int_0^1 \frac{1 - (1-x)^n}{x} dx = \int_0^1 (1 + (1-x) + \dots + (1-x)^{n-1}) dx = \sum_{k=1}^n \frac{1}{k}.$$

bor

(k^*) $f_0(1)$ qiymati $m = 0$ da kerakli summani hosil qiladi, bunda $f_0(x) = (x-1)^n$.

$f_m(x)$ ni rekursiy aniqlaymiz: $f_m(x) = x f_{m-1}'(x)$. Keyin $f_m(1) = 0$ va $m \leq n$ uchun kerakli summa. Bu yerdan xulosa qilish oson. $f_m(1) = 0$ uchun $m < n$ va $f_n(1) = n!$

(l^*) Tenglik $f(x) = \sum_{n \geq 0} C_{2n}^n x^n$ da $f(x)^2 = \frac{1}{1-4x} = \sum_{n \geq 0} 2^{2n} x^n$ ga teng.

Ikkinchisi $f(x) = (1-4x)^{-1/2}$ tenglikdan kelib chiqadi, funksiyani kengaytirish orqali tekshirish oson $(1-4x)^{-1/2}$ tenglikdagi Teylor qatori $x = 0$ nuqta.

(m^*) Da

$$\sum_{m=1}^k \frac{1}{m} = \int_0^1 (1+x+\dots+x^{k-1}) dx = \int_0^1 \frac{1-x^k}{1-x} dx$$

dan boshlab quyidagiga ega bo'lamiz

$$\sum_{k=1}^n (-1)^{k-1} C_n^k \sum_{m=1}^k \frac{1}{m} = \int_0^1 \frac{1 + (1-x)^n - 1}{1-x} dx = \frac{1}{n}.$$

Mustaqil yechish uchun masalalar

№ 1. (n) (Арнольд) C_n^k ekanligini isbotlang, agar k va n o'zaro teng bo'lsa, n ga bo'linadi ($k < n$)

(m) p tub son bo'lsin. $C_{2p}^p = 2 \pmod{p}$ ni ko'rsating.

№ 2. N sonining bo'limlari soni ko'pi bilan ikkita ekanligini ko'rsating, $\left[\frac{N}{2} \right] + 1$ ga teng, bu erda $\left[x \right] - x$ sonining butun qismi.

№ 3. Ikkinchi turdagi Stirling raqami S_N^n ta elementlar to'plamining mumkin bo'lgan qismlari soni sifatida aniqlanadi, aytaylik $1, 2, \dots, N$, N ta bo'sh bo'lmagan kichik to'plamlarga ega.

$1 \leq n \leq N$ da quyidagi xususiyatlar mavjudligini ko'rsating:

(a) $S_N^1 = S_N^N = 1$, $S_N^2 = 2^{N-1} - 1$, $S_N^{N-1} = C_N^2$;

(b) $S_{N+1}^n = S_N^{n-1} + nS_N^n$;

(c) $S_{N+1}^n = \sum_{k=0}^N C_N^k S_k^{n-1}$ $S_k^p = 0$ agar $p > k$;

(d) $S_N^n = \frac{1}{n!} \sum_{k=0}^n (-1)^{n-k} C_n^k k^N$;

(e) $S_N^n = \frac{1}{n!} \sum_{k=0}^n (-1)^k C_n^k (n-k)^N$;

(f) $S_N^n \leq n^{N-n} C_{N-1}^{n-1}$.

FOYDALANILGAN ADABIYOTLAR.

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